Comparison to Datalog/FOPC

- Semantics of FOPC can be thought of as possible worlds
  - Intended interpretation is $w_0$
  - All models of the KB are accessible from $w_0$: $w_0 R w_{\prime}$
  - Including the intended interpretation: $w_0 R w_0$

- For modal logic
  - There is a real world $w_0$ that the agent knows it is in (assumed to have complete knowledge)
  - Accessibility relation is not just from the initial world
  - Will allow us to model nested modal operators: $\text{bel(bel(raining))}$
  - Might not include $w_0 R w_0$
  - Agents might have incorrect beliefs

Worlds (interpretation) can be repeated, but with different accessibility relations

Modal Logic

- Modal operator operates on formulas (sentences in a logic)
- Modality is used to express: beliefs
- Can be used for expressing possibility:
  - $\text{possibly(raining)}$
- $\text{raining} \rightarrow \text{wet}$

- Can be used for expressing desires (user's goals)
- Want $m (\text{cleandishes})$
- $\text{cleandishes} \rightarrow \text{cleaningthem}$
- Can be used for expressing what the system can reason about:
  - Syntactically sound and complete
  - Grounded (by design)

Modal operator applies to formulas (sentences in a logic)
Distribution Axiom

\[ (\phi \land B\psi) \rightarrow (B\phi \land B\psi) \]

- This can equivalently be written as
- Distribution over implication
- This can equivalently be written as
- Possible worlds semantics forces agents to believe all consequences of their beliefs
- This is forced by the possible worlds semantics
- So agents believe all the consequences of their beliefs
- If \( B\phi \) and \( B(\phi \rightarrow \psi) \) does \( B\psi \)?

Epistemic Necessitation

- Does the agent know all necessary truths?
- Example:
  - \( \neg(\phi \land \psi) \leftrightarrow (\neg\phi \lor \neg\psi) \)
  - So it is true in every accessible world from any world
  - Possible worlds semantics is forcing us to make all tautologies true of modal operators
  - Can be written as: If \( \phi \) then \( B\phi \)
  - This is not the same as \( \phi \rightarrow B\phi \)
  - That is saying that the agent believes everything that is true in the world

Overview

- Omniscience
- Model Operator Axioms
- Syntax and Semantics
- Quantified Proofs
- Quantifiers and Quantifiers
- Model Operator Axioms
- Quantifiers
- Quantified Proofs
- Axioms
- Операции
Knowledge Axiom

- \( B\phi \rightarrow \phi \)
- Everything that user believes is true

Which is why it is called the knowledge axiom

- Not usually used for belief, but is used for knowledge

Consider if \( R \) is reflexive:

- \( wRw \) is true for every world

- Means that \( B\phi \) is only true if \( \phi \) is true in the current world

- Person’s beliefs (+ve and -ve) are subset of what is true in current world

- So if \( R \) is reflexive then \( B\phi \) is true for every world

- Which is why it is called the knowledge axiom

- Everything that user believes is true

\[ \phi \rightarrow B\phi \]

Properties on \( R \)

- \( R \) is euclidean if \( R \) has what two properties?

- Serial: for all \( w \) there exists \( w' \) s.t. \( wRw' \)

- Transitive: if \( wRw' \) and \( w'Rw'' \) then \( wRw'' \)

- Symmetric: if \( wRw' \) then \( w'Rw' \)

- Reflexive: for all \( w \), \( wRw \)

- Certain properties depend on what \( L \) is modeling, you might want \( R \) to have:

- Certain properties

Overview

- Constants and Quantifiers

- Syntactic Proofs

- Model Operator Axioms

- Common Associativity

- Common Associativity
Different Combinations

- Axioms:
  - Epistemic Necessitation
  - Distribution Axiom
  - Knowledge Axiom
  - Positive Introspection Axiom
  - Negative Introspection Axiom

- Can pick and choose which ones you want for your logic.

Modal Logic Constraints on R Axioms in Proof Theory

K: no constraint
D: reflexive
T: reflexive, transitive
S4: reflexive, transitive
S5: reflexive, transitive, symmetric
Weak S4: transitive
Weak S5: transitive, euclidean

Negative Introspection Axiom

- \( \neg B \phi \rightarrow B (\neg B \phi) \)
- If you do not believe something, you believe you do not believe it.

Positive Introspection Axiom

- \( B \phi \rightarrow B (B \phi) \)
- If a user believes something, the user believes they believe it.

Proof?

- Positive Introspection Axiom holds.
- If \( B \phi \) is true, then \( \neg B \neg \phi \).
- If \( B \phi \) is not true, someone else believes it.
The king tells two wise men that at least one of them has a white spot on his forehead. Each man can see the other's forehead but not his own. The first wise man says "I don't know whether I have a spot." The second says ...

- For a
  - The first wise man says "I don't know whether I have a spot." This means that either a or b has a spot. If a does not have a spot, b would know this, and b would hence know that a has a spot.
  - If a does not have a spot, b would know this, and b would hence know that a has a spot.

- For b
  - The second wise man knows whether a has a spot and that either a or b has a spot. If a does not have a spot, b would know this, and b would conclude that b has a spot, and b would answer "I know I have a spot." But he didn't, so b must know that a has a spot.

- For a
  - a knows that b knows whether a has a spot and that either a or b has a spot. a knows whether a has a spot and that either a or b has a spot.
  - a knows whether a has a spot and that either a or b has a spot.

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**Example Proof**

Nora believes \( p \rightarrow q \)

Nora does not believe \( q \) (she believes \( \neg q \) or has no belief about \( q \))

Show that Nora does not believe \( p \)

\( \neg \text{believes } p \rightarrow \neg \text{believes } q \)
Pitfall of Syntactic Proofs

- Extra axioms
- Converting to and from CNF
- What do we need to make it complete?
- Can we search effectively in this space?

Proving that $s(a) \rightarrow K_b((\neg s(a)))$ given:
1. $K_b(s(a) \lor s(b))$
2. $\neg K_b(s(b))$
3. $s(a) \lor K_b((\neg s(a)))$

CNF of 1

Can we rewrite 2 as $K_B(s(a)) \lor K_B(s(b))$? No.

$K_b((\neg s(a)) \rightarrow K_b(s(b)))$ Distribution Rule

CNF of 6

$s(a) \lor K_b(s(b))$ Resolution 4,7

$s(a)$ Resolution 3,8

Writing the Knowledge

- Says two wise men, so let’s view this as knowledge
- $a$’s knowledge
  - No need to put inside of a modal operator
  - $a$ or/and $b$ has a spot
- $a$’s knowledge of $b$’s knowledge
  - If $b$ knows $a$ or/and $b$ has a spot
  - $b$ knows whether $a$ has a spot
  - $b$ does not know whether $a$ has a spot
  - if $a$ has a spot, $b$ will know it
  - if $a$ does not have a spot, $b$ will know it
Overview

- Omniscient Proofs
- Model Operator Axioms
- Omniscient Proofs

Example

\[ \neg s(a) \rightarrow K_b (\neg s(a)) \]
\[ \neg s(a) \rightarrow (\forall w \ R(w,0) \rightarrow \neg T(s(a),w)) \]
\[ K_b (s(a) \lor s(b)) \]
\[ \neg K_b (s(b)) \]

Alternative: Proofs in Possible Worlds

- \( B(\alpha) \) in \( w \) means \( \alpha \) is true in all \( w' \) such that \( wRw' \)
- Let's use \( T(\alpha,w) \) to mean \( \alpha \) is true in \( w \)
- Usually use \( T \) and \( R \) to be predicates in FOPL logics
- \( T \) and \( R \) are used in our semantics to give meaning to the modal operator
- Worlds could then be objects in that language
- Predicates in modal logic become functions
- Predicates in modal logic become functions
- Worlds could then be objects in that language
- However, could view \( T \) and \( R \) to be predicates in FOPL logic
- Rather than use axioms, use corresponding restrictions on \( T \) and \( R \)
Constants and Quantifiers

• $B_j(\text{woman(mary)})$
  - Does the system and john agree on who mary is?

$\text{woman(mary)} \land B_j(\text{woman(mary)})$
  - Now do they agree?

• $\exists x B_j(\text{woman}(x))$ versus $B_j(\exists x \text{woman}(x))$
  - Does the system and john agree on who mary is?

Does the system and john agree on who mary is?