Deduction

• Not in book

1. Suppose to show there exists a DFA that recognizes \( L \)
2. Suppose to show that \( L \) is regular

Let \( L \) be any regular language

Prove that regular languages are closed under complementation

Show something is equivalent to a different problem

Prove that regular languages are closed under complementation

Let \( L \) be any regular language

Sufficient to show that \( \overline{L} \) is regular

Sufficient to show there exists a DFA that recognizes \( \overline{L} \)

So there must exist a DFA \( M \) that recognizes \( L \)

From a set of facts, deduce something that must be true

Closure Properties of Regular Languages

• Induction
• Construction
• Contradiction
• Deduction

Overview
Overview

- Deduction
- Construction
- Contradiction
- Induction
- Closure Properties of Regular Languages

How Much Detail?

- How is current line derived?
  - You can label each line and show what is used in deriving new lines
  - If obvious, or just use the previous line, don't need to number them
  - You can label each line and show what is used in deriving new lines
  - You don't have to be super detailed

If obvious, or just use the previous line, don't need to number them.
Regular Languages Closed under Complementation

Let \( L \) be any regular language.

Sufficient to show that \( \overline{L} \) is regular.

Sufficient to show there exists a DFA that recognizes \( \overline{L} \).

Since \( L \) is regular, there exists a DFA that recognizes it. Call it \( M \).

Let \( M = (Q, \Sigma, \delta, q_0, F) \).

Construct DFA \( M' = (Q, \Sigma, \delta, q_0, F') \) where

\[ F' = Q - F. \]

Claim:

\( L(M') = \overline{L} \).

Let \( w \in L \). So \( M \) accepts \( w \). Let \( q \) be the state that \( M \) is in at the end of processing \( w \). So \( q \in F \).

So when \( M' \) processes \( w \), at the end of processing \( w \), it will also be in \( q \).

Since \( q \not\in F' \), \( M' \) does not accept \( w \). So \( w \not\in L(M') \).

Similarly, if \( w \in L(M') \), then \( w \not\in L \).

So \( L(M') = \overline{L} \).

Proof by Construction

- Sometimes disguised as "Prove that all regular languages are closed under complementation.
- Prove that there exists a DFA that recognizes \( \overline{L} \).
- Show that there exists a DFA that recognizes \( L \).
- \( a \ast b \ast\) is regular. Show that a DFA that recognizes \( a \ast b \ast\) exists.
- \( a \ast b \ast\) is regular iff there exists a DFA that recognizes \( a \ast b \ast\).
- \( a \ast b \ast\) is regular. Prove by coming up with an example.
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Overview

- Deduction
- Construction
- Contradiction
- Induction
- Closure Properties of Regular Languages

How Much Detail?

- How much detail is needed?
  - Need to convince me that you know how to do the proof
  - Should be clear what you need to prove (2nd and 3rd line)
- Do you need to show construction does as is intended?
  - Should be clear when you need to prove (and not just state)
- Need to convince me that you know how to do the proof
- How much detail is needed?
  - Could add in:

\[ \text{So if } w \in L, \text{ then } w \not\in L'(M') \]

- Notice the word 'let', as in 'let \( x \) be ...'
- Sometimes you can handwave
- Is the last line obvious enough?
  - Could add in:

\[ \text{So } L = L'(M') \]
Proof by Contradiction

- Sometimes this is easier than trying to directly prove \( X \).

Assume that \( X \) is false, and show that you get a contradiction.

If you have to prove \( X \),
Induction

• Prove that all Xs have a certain property where all Xs can be categorized in terms of some property $P$ based on the natural numbers, such as:
  - number of states in a DFA
  - number of nodes in a graph
  - number of states in a DFA that are numbered in terms of some property $P$ based on the natural numbers

• Break problem into a base case and an induction step:
  - Base case: prove that $P(1)$ is true
  - Induction step: prove that if $P(i)$ is true for $i \geq 1$ then so is $P(i+1)$

After proving both parts, you know that $P(i)$ must be true for $i \geq 1$.

Prove that all X's have a certain property where all X's can be categorized in terms of some property $P$ based on the natural numbers, such as:

Induction

Closure Properties of Regular Languages

- Induction
- Contradiction
- Construction
- Deduction

Overview
Prove that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \) for \( n \geq 1 \).

**Proof by Induction:**

1. **Base case:** prove for \( n = 1 \)
   
   \[ \text{LHS} = \sum_{i=1}^{1} i = 1 \]
   
   \[ \text{RHS} = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1 \]

   Since \( \text{LHS} = \text{RHS} \), true for base case.

Continued

- Many variations of this:
Overview

Close Properties of Regular Languages
- Induction
- Construction
- Contradiction
- Induction

⇒

Induction step:
Assume it is true for \( n \), prove it is true for \( n + 1 \).

So assuming \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \),

Need to prove that \( \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} \).

Induction step: Assume it is true for \( n \), prove it is true for \( n + 1 \).
Proof Idea (continued)

• Approach 1:
  - We could have \( M \) first simulate \( M_1 \) and then simulate \( M_2 \)
  - This won't work because in simulating \( M_1 \), we will have used up the input with \( w \) on it, and we won't be able to get it back.

• Approach 2:
  - Simulate both \( M_1 \) and \( M_2 \) at the same time!
  - Have the states of \( M \) be the product of states of \( M_1 \) and \( M_2 \)
  - Have the transition function for \( M \) on \( w \) transition to the new state depending on what \( M_1 \) and \( M_2 \) would have individually done.
  - Have accepting states of \( M \) be any state with an accepting state from \( M_1 \) or from \( M_2 \).

Closed under Union

Definition of Union:
\[ A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \]

Class of regular languages is closed under union:
If a FA \( A \) accepts \( A \), and \( A \) accepts \( B \) (each accepted by a FA), then there is a FA that accepts \( A \cap B \) (class of regular languages).
Example Continued

- Draw the two states vertically.
- Draw a state diagram for $M_1$ with 3 states: $a$, $b$, $c$.

Let $A_1$ be strings that have an even number of 1's.

- Draw the 3 states horizontally.
- Draw a state diagram for $M_1$ with 3 states: $a$, $b$, $c$.

Let $A_2$ be strings in which the number of 0's is divisible by 3.

\[ \{ 0, 1 \} \]
Other Operations

- **Intersection**: \( A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \)
- **Complementation**: \( A^c = \{ x \mid x \notin A \} \)
- **Concatenation**: \( A \cdot B = \{ xy \mid x \in A \text{ and } y \in B \} \)
- **Star**: \( A^* = \{ x_1 x_2 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in A \} \)

**Formal Proof**

Let \( M_1 \) recognize \( A_1 \) where
\[
M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)
\]

Let \( M_2 \) recognize \( A_2 \) where
\[
M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)
\]

\[Q\]

- \( \delta \):
- \( q_0 \):
- \( F \):

Construct \( M \) to recognize \( A_1 \cup A_2 \) where
\[
M = (Q, \Sigma, \delta, q_0, F)
\]

Let \( N \) recognize \( A \) where
\[
N = (Q, \Sigma, \delta, q_0, F)
\]

Construct \( M \) to recognize \( A \cap B \) where
\[
M = (Q, \Sigma, \delta, q_0, F)
\]

Let \( N \) recognize \( A \) where
\[
N = (Q, \Sigma, \delta, q_0, F)
\]