Divide and Conquer

**Divide** the problem into a number of subproblems that are smaller instances of the same problem.

**Conquer** the subproblems by solving them recursively. If the subproblem sizes are small enough, just solve the subproblem in a straightforward manner.

**Combine** the solutions to the subproblems into the solution for the original problem.

- Two cases:
  - Recursive case
  - Base case

Overview

⇒ Chapter 4: Divide and Conquer
- Chapter 15: Dynamic Programming
- String Alignment Problems
- Framing the Problem Mathematically
- Algorithm for String Alignment
Simple Example of Divide and Conquer

• Membership testing in a sorted array: \( x \in L? \)
  + See if \( x \) is equal, less, or more than the element halfway in L
  + If less, test \( x \) with first half of L
  + We either find the element, or get to an empty array: conquer
  + Combine by passing answer back up the recursion
  + Takes \( O(\log(n)) \)

• Printing nodes in a tree
  - get str for left tree, for node, and for right tree
  - at a leaf, return string of node: conquered
  - combine: create string for entire node

Recursion versus Iteration

• Which of these two methods can be done using Iteration?

```python
def InOrderWalk(self):
    if self.left is not None:
        self.left.InOrderWalk()
    print self.key
    if self.right is not None:
        self.right.InOrderWalk()

def Search(self, k):
    if k == self.key:
        return self
    if k < self.key and self.left is not None:
        return self.left.Search(k)
    if k > self.key and self.right is not None:
        return self.right.Search(k)
    return None
```
Recursions

- recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs
  - natural way to characterize running time of divide and conquer algorithm
- Chapter 12: Printing tree
  - Had a recursive algorithm
  - Running time expressed as a recurrence
    \[ T(n) \leq \begin{cases} \frac{c}{T(k)} + T(n - k - 1) + d & n > 1 \\ c & n = 1 \end{cases} \]
  - We used ‘substitution method’ to solve this
    + Guess the solution
    + Use induction to prove solution is correct
  - Textbook gives two other methods for solving them

Recursion

- If problem is divided into one smaller problem, can use loop
  - Membership testing (pick one half of the list)
- If problem is divided into several problems, use recursion
  - Printing tree, need to print both sides
Technicalities in Recurrences

- Emphasis on recurrences for large values of $n$
  - Ignore differences due to odd or even input size
  - Ignore differences for boundary conditions (on small $n$)
    - Will only effect running time by a constant, which is irrelevant for $O$ and $\Theta$

Example: Maximum-Subarray Problem

- Find the biggest upshift in prices
  - Perhaps to see how well you did in a trading a stock versus the optimum
  - Can just buy/sell at end of day, over fixed period of time (say 100 days)
  - Can hold onto stock for any number of days
  - Must buy stock before selling it (no short sales)
Naive Solution

- Find global min and max: but global max might be before global min
- Other solutions?

Brute force

- Look at every pair of dates to find best one
- array price has the end-of-day prices
- Running time $\Theta(n^2)$

```python
best = -1
for j in range(99):
    for i in range(j+1,100):
        gain = price[i] - price[j]
        if gain > best:
            best = gain
```
Key Insight

- Rather than focus on the daily price
  - Focus on how much price has changed since prior day
  - Let \( \text{delta}(i) = \text{price}(i) - \text{price}(i-1) \)
  - Find a nonempty continuous subarray whose values have the largest sum
    + Referred to as 'Maximum subarray'

<table>
<thead>
<tr>
<th>Day</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>100</td>
<td>113</td>
<td>110</td>
<td>85</td>
<td>105</td>
<td>102</td>
<td>86</td>
<td>81</td>
<td>63</td>
<td>81</td>
<td>101</td>
<td>94</td>
<td>106</td>
<td>101</td>
<td>94</td>
<td>90</td>
<td>97</td>
</tr>
<tr>
<td>Change</td>
<td>13</td>
<td>-3</td>
<td>-25</td>
<td>20</td>
<td>-3</td>
<td>-16</td>
<td>-23</td>
<td>18</td>
<td>20</td>
<td>-7</td>
<td>12</td>
<td>-5</td>
<td>-22</td>
<td>15</td>
<td>-4</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

- For dividing the problem
  - Split delta array in half
  - Max subarray is either entirely on one side or spans halfway point

Spanning Midpoint

- Maximum subarray that spans both sides (includes \text{delta(mid)} and \text{delta(mid+1)})
  - First, go backward from \text{mid} and find max sum
  - Then, go forward from \text{mid+1} and find max sum
  - Can be done in \( \Theta(n) \)
  - Not smaller instance of original problem, as it has an added restriction

```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
1  left-sum = -\infty
2  sum = 0
3  for i = mid downto low
4      sum = sum + A[i]
5      if sum > left-sum
6          left-sum = sum
7      max-left = i
8      right-sum = -\infty
9      sum = 0
10     for j = mid + 1 to high
11        sum = sum + A[j]
12        if sum > right-sum
13          right-sum = sum
14      max-right = j
15  return (max-left, max-right, left-sum + right-sum)
```
Find Max Subarray

\[\text{FIND-MAXIMUM-SUBARRAY}(A, \text{low}, \text{high})\]

1. if \(\text{high} = \text{low}\)
2. return \((\text{low}, \text{high}, A[\text{low}])\) \(\text{// base case: only one element}\)
3. else \(\text{mid} = [(\text{low} + \text{high})/2]\)
4. \((\text{left-low}, \text{left-high}, \text{left-sum}) = \text{FIND-MAXIMUM-SUBARRAY}(A, \text{low}, \text{mid})\)
5. \((\text{right-low}, \text{right-high}, \text{right-sum}) = \text{FIND-MAXIMUM-SUBARRAY}(A, \text{mid} + 1, \text{high})\)
6. \((\text{cross-low}, \text{cross-high}, \text{cross-sum}) = \text{FIND-MAX-CROSSING-SUBARRAY}(A, \text{low}, \text{mid}, \text{high})\)
7. if \(\text{left-sum} \geq \text{right-sum} \text{ and } \text{left-sum} \geq \text{cross-sum}\)
8. return \((\text{left-low}, \text{left-high}, \text{left-sum})\)
9. elseif \(\text{right-sum} \geq \text{left-sum} \text{ and } \text{right-sum} \geq \text{cross-sum}\)
10. return \((\text{right-low}, \text{right-high}, \text{right-sum})\)
11. else return \((\text{cross-low}, \text{cross-high}, \text{cross-sum})\)

• Divide problem, conquer each part, combine

Running Time

• Let \(T(n)\) be running time of algorithm on input size of \(n\)
  - For simplicity, assume \(n\) is a power of 2

\[\text{FIND-MAXIMUM-SUBARRAY}(A, \text{low}, \text{high})\]

1. if \(\text{high} = \text{low}\)
2. return \((\text{low}, \text{high}, A[\text{low}])\) \(\text{// base case: only one element}\)
3. else \(\text{mid} = [(\text{low} + \text{high})/2]\)
4. \((\text{left-low}, \text{left-high}, \text{left-sum}) = T(n/2)\)
5. \((\text{right-low}, \text{right-high}, \text{right-sum}) = T(n/2)\)
6. \((\text{cross-low}, \text{cross-high}, \text{cross-sum}) = \Theta(n)\)
7. if \(\text{left-sum} \geq \text{right-sum} \text{ and } \text{left-sum} \geq \text{cross-sum}\)
8. return \((\text{left-low}, \text{left-high}, \text{left-sum})\)
9. elseif \(\text{right-sum} \geq \text{left-sum} \text{ and } \text{right-sum} \geq \text{cross-sum}\)
10. return \((\text{right-low}, \text{right-high}, \text{right-sum})\)
11. else return \((\text{cross-low}, \text{cross-high}, \text{cross-sum})\)

Lines 1-3: \(T(1) = \Theta(1)\)

Lines 7-11: constant time
Running Time

• Let $T(n)$ be the running time of the algorithm on input size of $n$
  - Base case (just one element)
    + $T(1) = \Theta(1)$ (Lines 1 to 3 take constant time)
  - Recursive step: ($n > 1$)
    + Lines 1-3 take $\Theta(1)$
    + Line 4 takes $T(n/2)$
    + Line 5 takes $T(n/2)$
    + Line 6 takes $\Theta(n)$
    + Line 7-11 take constant time
    \[
    T(n) = 2T(n/2) + \Theta(n) + \Theta(1)
    = 2T(n/2) + \Theta(n)
    \]

Solving the Recurrence

• Used substitution method previously
  - Guess the form, and prove by induction
    - Works for $O$ (upper bound), but not for $\Theta$
• Master theorem: Another way of solving recurrences
  - Cookbook method for recurrences of form $T(n) = aT(n/b) + f(n)$
    where $a \geq 1$, $b > 1$ and $f(n)$ is asymptotically positive function
  - Captures any algorithm that divides problem into $a$ smaller ones of size $n/b$, and solves them recursively
  - Technical point: More correct to use $T(\lfloor n/b \rfloor)$, but will not affect the asymptotic behavior
Master Theorem

Theorem 4.1 (Master theorem)
Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret $n/b$ to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^\log_b a - \epsilon)$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^\log_b a)$.
2. If $f(n) = \Theta(n^\log_b a)$, then $T(n) = \Theta(n^\log_b a \log n)$.
3. If $f(n) = \Omega(n^\log_b a + \epsilon)$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$, then $T(n) = \Theta(f(n))$. ■

So for $T(n) = 2T(n/2) + \Theta(n)$

$+ a = 2, b = 2, f(n) = \Theta(n) = \Theta(n^1) = \Theta(n^\log_2 2)$

So use second case: $T(n) = \Theta(n \log n)$

Essential Point of Divide and Conquer

- **Optimal Substructure:**
  - solution to a given optimization problem can be obtained by the combination of optimal solutions to its subproblems

- **Non-overlapping subproblems:**
  - There is an obvious way to break the problem into subproblems (do not have to search over different subproblems)
  - The division into subproblems will give you the optimal solution
Problem not solvable with Divide and Conquer

• Finding the best route from A to B
• Divide problem,
  - Unclear what C should be used to split problem into A to C, and C to B
  - Many different ways to divide into subproblems
  - Choice will affect whether you find the optimal solution

Overview

• Chapter 4: Divide and Conquer
  ⇒ Chapter 15: Dynamic Programming
• String Alignment Problems
• Framing the Problem Mathematically
• Algorithm for String Alignment
Dynamic Programming

- Similar to divide-and-conquer
- Optimal Substructure:
  - Solution to a given optimization problem can be obtained by the combination of optimal solutions to its subproblems
- Overlapping subproblems:
  - There is not a single obvious way to break the problem into subproblems

Example: Routing

- Has the optimal substructure problem
  - If C is on the optimal path from A to B (so A to B can be optimally divided into subproblems A to C and C to B)
    + Optimal solution to A to B is: optimal solution from A to C followed by optimal solution from C to B
- Overlapping subproblems:
  - Route from A to B can be divided by going through C, or D, or E
    + Don’t know which division is best
Key Insight into Efficient Solution

- You have overlapping subproblems
e.g., A to C and C to B; versus A to D and D to B
  - there might be subproblems in common between these subproblems
    e.g., A to X might be used in A to C and A to D
    e.g., X to Y might be used in A to C and A to D and C to B and D to B
- Save solutions to these subproblems and do not recompute!
  - Memoize the results (or store in a table)
  - ‘Programming’ in dynamic programming actually refers to storing intermediate results in a table

Brute Force

- Try each possible partition at each level
- Could lead to exponential time algorithm
Top Down or Bottom Up

- In a top down implementation
  - Before doing a subproblem, check table to see if already done it
- In bottom up
  - Start with small problems, and build up to larger ones
  - More straightforward

Overview

- Chapter 4: Divide and Conquer
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Matching DNA Sequences

- DNA of one organism might be:
  ACCGGTCTGAGTCGGCGGAAAGCCGGC

- Of another:
  GTCTCGGAAATGCCGGTTGCTCTGTAAA

- How similar are the strands?
  - What sequence of bases are common in both strands?
  - Find an alignment with the maximum number of matches

```
ACCGGTTAGTCGCGGAA GCCG GC C G AA
   GTCT CTGG TCCG TAA
   GTCG TT CG GAATTGC CGTTGCTCTGTAAA
   GTCT C TCG GAA GCCG GC C G AA
```
Sentence Recall Test

- Examiner says a sentence to the participant:
  the little boy went to the store
- Participant tries to repeat it verbatim:
  the boy went to the store
  the boy went to a store
  a little lad goed to a store
- Align the two sentences to find what the mistakes are
  - Interested in insertions, deletions, and substitutions
  the → a
  boy → lad
  little → ε
  went → goed

Overview

- Chapter 4: Divide and Conquer
- Chapter 15: Dynamic Programming
- String Alignment Problems
  ⇒ Framing the Problem Mathematically
- Algorithm for String Alignment
How do we frame the problem?

- For any alignment between two sequences
  - Score it based on the number of insertions, deletions, and substitutions
  - Penalty of 1 for insert/delete/sub and no penalty for match

- Find the alignment with the smallest penalty

- Let’s be more precise about what an alignment is
  - Let $a, b \in \Sigma^*$ and $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_m$
  - Which indexes of $a$ and $b$ are part of a match or substitution
    - $A \subset \{1, 2, 3, \ldots, n\}$ and $B \subset \{1, 2, 3, \ldots, m\}$
    - $2^n$ possible values for $A$ and $2^m$ for $B$
    - $2^{n+m}$ possible alignments

Example

- Alignment was implicit in the formatting
  - Blank spots have an epsilon in there alignment

- One alignment (8 deletes, 6 inserts, 1 subs)
  \[
  \begin{array}{ccccccccc}
  A & C & C & G & G & T & C & G & A \\
  \end{array}
  \]

- A worse one (3 deletes, 2 inserts, 13 subs)
  \[
  \begin{array}{ccccccccc}
  A & C & C & G & G & T & C & G & T & C & G & G & A & G & C & G & A & G & T & C & G & T & C & G & T & C & G & T & A & A \\
  \end{array}
  \]

- Pick the best alignment (one with lowest score)
A Better Way to View an Alignment

- $a$ (prompt) has $n$ words/characters, $b$ (response) has $m$ words
- View prompt as columns and response as rows ($n \times m$ array)
- An alignment is a path through the cells where you can go:
  - Left (consume a word of the prompt but no word of response)
    + deletion
  - Down (consume a word of the response but no word of prompt)
    + insertion
  - Diagonal left-down (consume a word of response and prompt)
    + If words are the same: match, otherwise substitution

Example

<table>
<thead>
<tr>
<th></th>
<th>the</th>
<th>little</th>
<th>boy</th>
<th>went</th>
<th>to</th>
<th>the</th>
<th>store</th>
</tr>
</thead>
<tbody>
<tr>
<td>little</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>lad</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>good</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>to</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>store</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>yeah</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

X X
X
X X
X
X X
X
X
Overview

- Chapter 4: Divide and Conquer
- Chapter 15: Dynamic Programming
- String Alignment Problems
- Framing the Problem Mathematically
  ⇒ Algorithm for String Alignment

Brute Force

- How many paths through array are there?
- You can just go right, down or diagonally right down
  - Path is at most $n + m$ in length (no matches or substitutions)
  - At each point in path, at most 3 options (left, down, diagonal)
  - At most $3^{n+m}$ paths
  - For each path, compute score
    + For each left or right move, add 1
    + For each diagonal move: determine if it is a match (0) or substitution (1)
- Can do some optimization
  - Keep track of best path so far, and prune paths if they are worse
  - Can do this as a depth-first search
    + Gets rid of some redundancy (of the first parts of the path)
    + But bottom right hand corner will be redone many times
Dynamic Programming

- Optimal Substructure?
  - If (i,j) and (k,l) is in the optimal solution, optimal path from (i,j) to (k,l) is part of solution

- Overlapping subproblems
  - Optimal path from (i,j) to (k,l) can be a subproblem of a lot of larger problems

### Initialization

- Each cell will have optimal score to get from (0,0)
  - Fill in 0th row. Each move left implies we are deleting
  - Fill in 0th column. Each move down implies we are inserting

<table>
<thead>
<tr>
<th></th>
<th>the</th>
<th>little</th>
<th>boy</th>
<th>went</th>
<th>to</th>
<th>the</th>
<th>store</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>little</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>lad</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>goed</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>to</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>store</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>yeah</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Filling in cells

- Fill in cell \((i, j)\) if cells above \((i, j-1)\), left \((i-1, j)\), diag \((i-1, i-j)\) filled in
- 3 possible ways to get to cell
  + From above: take score \((i, j-1)\) and subtract 1 (insert)
  + From left: take score of \((i-1, j)\) and subtract 1 (delete)
  + From left-above: take score of \((i-1, j-1)\) if match, else subtract 1 (substitution)
  + Take lowest

|        | the | little | boy | ...
|--------|-----|--------|-----|-----
| 0      | 1   | 2      | 3   | ...
| little | 1   | 1      |     |     
| lad    | 2   |        |     |     
| goed   | 3   |        |     |     
| ...    | ... |        |     |     

Efficiency?

- Initialize row 0 \(\Theta(n)\)
- Initialize col 0 \(\Theta(m)\)
- For \(m \times n\) cells
  - Determine value of each cell \(\Theta(nm)\)
- Assume \(n\) and \(m\) are similar in size
  - \(\Theta(n^2)\) rather than \(\Theta(3^n)\)
How is it a Dynamic Programming Solution

```python
for i in range(1,m):
    for j in range(1,n):
        diff = 0 if prompt[i] == response[j] else 1
        cell[i,j] = max(cell[i-1,j]+1,
                        cell[i,j-1]+1,
                        cell[i-1,j-1]+diff)
```

• Bottom-up algorithm
  + Order the subproblems from smallest to biggest
  + Will already have values for smaller problems when needed by bigger problems

• Dynamic Programming
  + `cell[i,j]` optimal score to get to `(i,j)`
  + `cell[i,j]` calculated just once!
  + `cell[i,j]` used to calculate `cell[i+1,j]` `cell[i,j+1]` `cell[i+1,j+1]`
  + Not just 3 times as fast but changing it from $3^n$ to $n^2$